

Continuous Flattening of Extended Bipyramids with Rigid Radial Edges

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Abstract

Can we flatten a polyhedron of flexible material such as a piece of paper without cutting and stretching? If the polyhedron is convex, it has been proved that there are infinitely many ways to do so [1, 2], but if we assume that some parts of the polyhedron are rigid, we have a new version of the problem, and it is natural and important to set conditions that will be appropriate for applications in industry. In this talk, we define a semi-bipyramid and show that any semi-bipyramid can be continuously flat-folded so that any radial edges are rigid. Furthermore, we define an extended bipyramid and give a similar result.

Definition 1. An n -gonal *bipyramid* is a polyhedron formed by joining an n -gonal pyramid and its mirror image base-to-base. An n -gonal *semi-pyramid* is a convex polyhedron with isosceles triangular faces whose bases comprise an n -gon inscribed in a circle. We assume the centre of the circle is included in the interior or the boundary. An n -gonal *semi-bipyramid* is a polyhedron formed by joining an n -gonal semi-pyramid and its mirror image base-to-base (see Figure 1). The edges incident at a peak of a semi-pyramid are called *radial edges*.

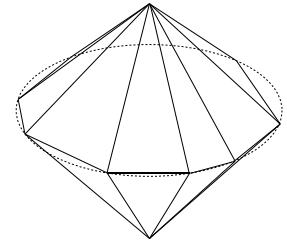


Figure 1: A semi-bipyramid

Since all triangular faces of a semi-bipyramid are isosceles triangles, the orthogonal projection of the peak to the plane with the base is the centre of circumscribed circle of the base.

Theorem 1. *Each n -gonal semi-bipyramid can be continuously flattened so that the radial edges are rigid, that is, they are not folded during the motion.*

Definition 2. Let F be a convex k -gon (v_1, v_2, \dots, v_k) and assume that the angles at both v_1 and v_k are not obtuse. An *extended bipyramid* is a polyhedron formed by rotating F n times around the edge $v_1 v_k$ so that the sum of rotated angles θ_i ($0 < \theta_i \leq \pi; i = 1, \dots, n$) is 2π , and taking the surface of the convex hull of all vertices (see Figure 2). We denote it by $EB(v_1, v_2, \dots, v_k; \theta_1, \theta_2, \dots, \theta_n)$ or $EB=EB(F, n)$. The edges which are images of ones in F are called *radial edges*, and the rotated angles are called *sector angles*. Divide EB by the sector angles into n pieces, and call these *sector pieces*. The polyhedron formed by a sector piece together

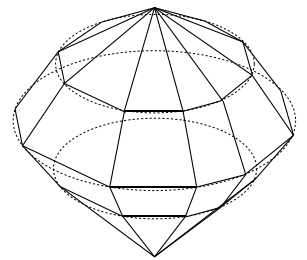


Figure 2: An extended binvramid

with two faces congruent to F is called a *sector polyhedron*. Sector polyhedrons that satisfy the following condition (C), (as well as EB) are said to be *simple*.

Condition (C): Any inscribed sphere touching two adjacent faces with a maximal radius touches both faces congruent to F .

If a closed sector piece S of EB is simple, the set of all points touching at least three faces of S , together with the edges of S , divide the surface of S ; and we call such division the *straight skeleton subdivision*. Taking the straight skeleton subdivision for all closed sector pieces, we get the straight skeleton subdivision of EB . Each non-radial edge of EB is a common edge of two reflectively congruent triangles.

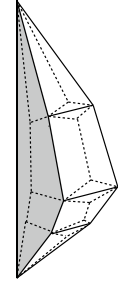


Figure 3: An example of the straight skeleton subdivision

Theorem 2. *Each simple extended bipyramid can be continuously flattened so that all radial edges are rigid, that is, they are not folded during the motion.*

The following kite property plays a key role in proofs.

Kite property: Let $K=abcd$ be a kite with $|ab| = |ad|$ and $|cb| = |cd|$, centre h , and q and q' be points of bh and dh , respectively, so that $|qh| = |q'h|$. Fold K by mountain creases along bh , aq' , cq' , and $q'd$, and valley creases along ah , ch , hq' so that the point q' attaches to the point q (Figure 4). We call the resulting figure a *folded kite with wing-shaped*. This figure is flexible. If the distances of two pairs of diagonal vertices of K in the resulting figure are given, such as *distance* $\{a, c\} = l$ and *distance* $\{b, d\} = m$, the resulting figure is uniquely determined up to congruence. We denote it by $K(l, m)$. For any pair $\{l, m\}$ with $||ah| - |ch|| \leq l \leq |ac|$ and $0 \leq m \leq |bd|$, there is a unique point q so that one of its folded kites is congruent to $K(l, m)$. For any two folded kites $K(l_1, m_1)$ and $K(l_2, m_2)$ there is a family $\{K(l_t, m_t) : 1 \leq t \leq 2\}$ such that $K(l_t, m_t)$ converges to $K(l_2, m_2)$ as t increases from one to two.

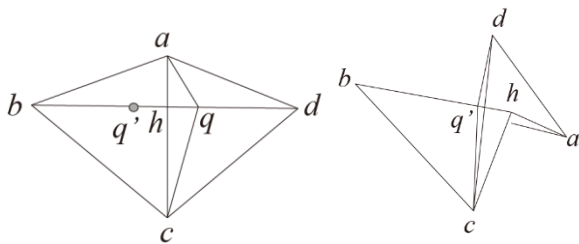


Figure 4: A folded kite with wing-shaped

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